On Some Measure of Conditional Uncertainty in Past Life

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Abstract: In the context of information theory, conditional measure of uncertainty in remaining life time has been proposed by P.G. Sankaran and R.P. Gupta (Calcutta Statistical Association Bulletin, Vol. 49; 159-166, (1999)). The conditional measure of uncertainty is very much useful in study of ageing pattern of the system. In this paper we have introduce a new measure of conditional uncertainty based on past life and we have study some properties of it. We develop a new ageing class of life distribution based on our new proposed conditional uncertainty and study property of it. We also have characterized *DRHR* and *IRHR* class of life distribution using our proposed new measure of uncertainty based on conditional expectation.

Keywords: Conditional uncertainty, Residual entropy, Past entropy, *IUL*, *ICUPL*, *DRHR*, *IRHR*, Reversed hazard rate function, Hazard rate function.

1. Introduction:

Let X be an absolutely continuous non-negative random variable having distribution function $F(t) = P(X \le t)$ and survival function $\bar{F}(t) = P(X > t)$. X may be the lifetime of a component/system or of a living organism. The basic measure of uncertainty is defined by Shannon (1948) as

$$H(f) = -E(\ln f(X)) = -\int_0^\infty f(x)\ln f(x)\,dx, \qquad \text{---------------- (1.1)}$$

where $f(x)$ is the density function of X. This is known as Shannon information measure. The properties and virtues of $H(f)$ have been thoroughly investigated since the work of Shannon (1948) and Weiner (1961).

The random variable $(X-t)$ given $X > t$ represents the residual life of a unit with age t. In recent years the role of Shannon entropy as a measure of uncertainty in residual life distributions has been studied by many researchers (cf. Ebrahimi (1996), Ebrahimi and Kirmani (1996a, b), Ebrahimi and Pellerey (1995)).

Ebrahimi (1996) defined the uncertainty of residual lifetime distributions $H(f;t) = -E(\ln f_t(X))$, where $f_t(x)$ is the probability density function of $(X-t)$ given $X > t$. By writing $r_F(t) = \frac{f(t)}{\bar{F}(t)}$, the failure rate / hazard rate function of X , (1.2) can equivalently be written as

$$H(f;t) = -\int_t^\infty f_t(x)\ln f_t(x)\,dx$$
$$= -\int_t^\infty \frac{f(x)}{\bar{F}(t)}\ln\left(\frac{f(x)}{\bar{F}(t)}\right)dx$$
$$= 1 - \frac{1}{\bar{F}(t)}\int_t^\infty f(x)\ln r_F(x)\,dx. \qquad \text{-------------- (1.2)}$$

It is reasonable to presume that in many realistic situations uncertainty is not necessarily related to the future but can also refer to the past. For instance, if at time t, a system which is observed only at certain pre assigned inspection times, is found to be down; then the uncertainty of the system life relies on the past i.e., on which instant in $(0,t)$ it has failed. Based on this idea Di Crescenzo and Longobardi (2002) have studied the past entropy over $(0,t)$. They have discussed the necessity of the past entropy, its relation with residual entropy and many interesting results. If X denotes the life time of a component/ system or of living organism, then past entropy of X at time t is defined as

$$H^*(f;t) = -E(\ln f_t^*(X)) = -\int_0^t \frac{f(x)}{F(t)} \ln\left(\frac{f(x)}{F(t)}\right) dx, \qquad \text{-------------- (1.3)}$$

where $f_t^*(x)$ is the probability density function of the random variable $(t - X)$ given $X \le t$, which represents the past life of component/ system or of living organism survived up to an age t.

Based on residual life distribution Sankaran and Gupta (1999) have introduced a new measure of uncertainty known as conditional measure of uncertainty (CMU). They have discussed some properties and characterization results based on CMU. They also have shown a relation between CMU and residual entropy.

The conditional measure of uncertainty in residual life is defined as

$$M(f;t) = E(-\ln f(X)|X > t) = -\frac{1}{\bar{F}(t)}\int_t^{\infty} f(x)\ln f(x)dx. \qquad \text{-------------- (1.4)}$$

In this paper we have introduce a new kind of conditional measure of uncertainty based on past life and we have discuss some properties of it. We develop here a new ageing class $(ICUPL\ class)$ of life distribution based on our new proposed conditional uncertainty and study properties of it. In section 2, we have established a relationship between $ICUPL$ ageing class and ageing class IUL based on past entropy (see Nanda and Paul (2006)). We also have characterized decreasing reversed hazard rate $(DRHR)$ and increasing reversed hazard rate $(IRHR)$ class of life distribution using this new measure of uncertainty based on conditional expectation.

2. Some properties based on conditional measure of uncertainty:

In this section we have defined a new kind of conditional measure of uncertainty based on past life.

Definition 2.1: *For a non-negative random variable X, the conditional uncertainty based on past life is defined as $M^*(f;t) = E(-\ln f(X)|X \le t)$.* ■

$M^*(f;t)$ can also be written as $M^*(f;t) = -\frac{1}{F(t)}\int_0^t f(x)\ln f(x)dx.$ ------------------ (2.1)

$M^*(f;t)$ measures the uncertainty contained in $f(x)$ about the predictability of total lifetime of the unit which has survived up to an age t, while $H^*(f;t)$ gives the uncertainty contained in $f_t^*(x)$ about the predictability of the past lifetime of the unit.

It is easy to observe that

$$H^*(f;t) = -\frac{1}{F(t)}\int_0^t f(x)\{\ln f(x) - \ln F(t)\}dx$$

$$= -\frac{1}{F(t)}\int_0^t f(x)\ln f(x)\,dx + \frac{1}{F(t)}\int_0^t f(x)\ln F(t)\,dx$$

$$= \frac{\ln F(t)}{F(t)}\int_0^t f(x)dx - \frac{1}{F(t)}\int_0^t f(x)\ln f(x)\,dx$$

$$= \ln F(t) + M^*(f;t) \qquad \text{(by using (2.1))}.$$

Differentiating the above equation with respect to t we get

$$H^{*\prime}(f;t) = \frac{1}{F(t)}\frac{d}{dt}F(t) + M^{*\prime}(f;t) = \frac{f(t)}{F(t)} + M^{*\prime}(f;t)$$

$$= \mu_F(t) + M^{*\prime}(f;t), \qquad \text{-------------- (2.2)}$$

where $\mu_F(t) = \frac{f(t)}{F(t)}$ is the reversed hazard rate function of the random variable X.

Based on $H^*(f;t)$, Nanda and Paul (2006) have defined the following non-parametric classes of life distribution viz. increasing uncertainty of life (IUL) as:

Definition 2.2: A random variable X is said to have increasing uncertainty of life (IUL) if $H^*(f;t)$ is increasing in $t \geq 0$. ■

In this section, we propose a new type of non-parametric class of lifetime distributions based on $M^*(f;t)$.

Definition 2.3: *A random variable X is said to have increasing conditional measure of uncertainty on past life $(ICUPL)$ if $M^*(f;t)$ is increasing in $t \geq 0$.* ■

To see that not all distribution are monotone in terms of $M^*(f;t)$, we provide below an example.

Example 2.1: Let X be a non-negative random variable with distribution function

$$F(x) = \begin{cases} \frac{x^2}{2}, & if\ 0 \leq x \leq 1 \\ \frac{x^2+2}{6}, & if\ 1 \leq x \leq 2 \\ 1, & if\ x \geq 2. \end{cases}$$

Then, we have

$$M^*(f;t) = \begin{cases} \frac{1}{2} - \ln t, & 0 \leq t \leq 1 \\ \frac{1}{t^2+2}\left[\frac{3t^2}{2} - t^2 \ln\frac{t}{3} - \ln 3\right], & 1 \leq t \leq 2 \\ 1 - \frac{2}{3}\ln\frac{2}{3} - \frac{1}{6}\ln 3, & t \geq 2. \end{cases}$$

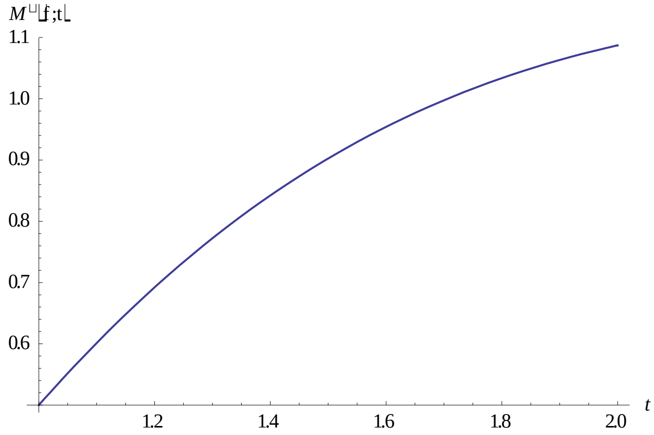

Figure 1: Plot of $M^*(f;t)$ (Example 2.1) against $t \in [1,2]$.

It is not very hard to check that $M^*(f;t)$ is decreasing when $0 \le t \le 1$ and $M^*(f;t)$ is constant when $t \ge 2$. But, from Figure 1, it is clear that $M^*(f;t)$ is increasing for all values of $t \in [1,2]$. Hence $M^*(f;t)$ is not monotone for all values of $t \ge 0$. ■

Now, we establish a relationship between IUL class and $ICUPL$ class of lifetime distributions.

Theorem 2.1: If X is $ICUPL$ then $X \in IUL$.

Proof: Let X be $ICUPL$, then $M^*(f;t)$ is increasing in $t \ge 0$. Thus $M^{*'}(f;t) > 0$.

From equation (2.2), we have $H^{*'} = \mu_F(t) + M^{*'}(f;t) > 0$, since $\mu_F(t) > 0$ for all $t \ge 0$. This implies that $X \in IUL$. Hence the proof. ■

Below is an example to show that IUL property does not imply $ICUPL$ property.

Example 2.2: Let X be a non-negative random variable with distribution function

$$F(x) = \begin{cases} \frac{x^2}{2}, & if\ 0 \le x \le 1 \\ \frac{x^2+2}{6}, & if\ 1 \le x \le 2 \\ 1, & if\ x \ge 2. \end{cases}$$

Then, we have

$$H^*(f;t) = \begin{cases} \ln\left(\frac{t}{2}\right) + \frac{1}{2}, & 0 \le t \le 1 \\ \ln\left(\frac{t^2+2}{6}\right) + \frac{t^2-1}{t^2+2}\ln 3 - \frac{t^2}{t^2+2}\ln t + \frac{1}{2}, & 1 \le t \le 2 \\ \frac{1}{2}\ln 3 - \frac{2}{3}\ln 2 + \frac{1}{2}, & t \ge 2, \end{cases}$$

That $H^*(f;t)$ is increasing in t is shown in Figure -2 where the graph is plotted for $t \in [0,2]$ and beyond this interval, $H^*(f;t)$ is constant. Further, from Example-2.1 it is clear that $M^*(f;t)$ is not monotone for all values of $t \ge 0$. Hence, X is IUL but not $ICUPL$.

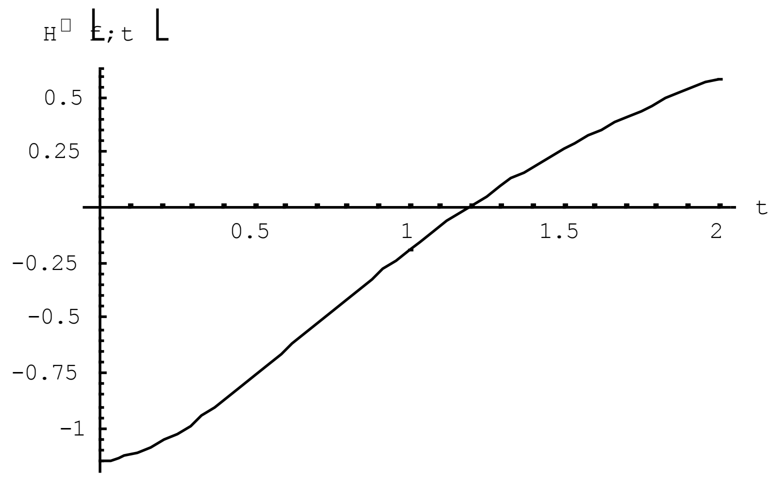

Figure 2: Plot of $H^*(f;t)$ (Example 2.2) against $t \in [0,2]$.

Now we have a characterization for decreasing reversed hazard rate $(DRHR)$ and increasing reversed hazard rate $(IRHR)$ classes of lifetime distribution using $M^*(f;t)$ and $H^*(f;t)$.

<u>Theorem 2.2</u>: (i) If X is $DRHR$ then $M^{*''}(f;t) \geq H^{*''}(f;t)$ for all $t \geq 0$ and (ii) If X is $IRHR$ then $M^{*''}(f;t) \leq H^{*''}(f;t)$ for all $t \geq 0$, where $M^{*''}(f;t)$ and $H^{*''}(f;t)$ denotes the second derivative of $M^*(f;t)$ and $H^*(f;t)$ respectively.

<u>Proof</u>: Differentiating (2.2) with respect to t, we get

$$H^{*''}(f;t) = \frac{d}{dt}(\mu_F(t)) + M^{*''}(f;t) \qquad \text{--------------- (2.3)}$$

(i) If X is $DRHR$, then $\mu_F(t)$ is decreasing in $t \geq 0$ *i.e.* $\frac{d}{dt}(\mu_F(t)) \leq 0$. Thus, from (2.3) we get $M^{*''}(f;t) \geq H^{*''}(f;t)$ for all $t \geq 0$.

(ii) If X is $IRHR$, then $\mu_F(t)$ is increasing in $t \geq 0$ *i.e.* $\frac{d}{dt}(\mu_F(t)) \geq 0$. Thus, from (2.3) we get $M^{*''}(f;t) \leq H^{*''}(f;t)$ for all $t \geq 0$. Hence the proof. ■

<u>Remark 2.1</u>: If X is IUL and $M^{*'}(f;t)$ is decreasing in $t \geq 0$, then X has $IRHR$ property.

The following proposition which gives the value of $M^*(f;t)$ under linear transformation, will be used in proving the upcoming theorem of this section.

<u>Proposition 2.1</u>: For any absolutely continuous random variable X, define $Z = aX$, where $a > 0$ is a constant. Then for $t \geq 0$,

$$M^*(f_z;t) = M^*\left(f;\frac{t}{a}\right) + \ln a\,,$$

where f_z is the density function corresponding to the random variable Z.

<u>Proof</u>: The distribution function $(F_Z(\cdot))$ and the density function $(f_Z(\cdot))$ of the random variable Z can be written as $F_Z(t) = F\left(\frac{t}{a}\right)$ and $f_Z(t) = \frac{1}{a} f\left(\frac{t}{a}\right)$, where $F(\cdot)$ and $f(\cdot)$ denote distribution and density function of the random variable X respectively.

Now,
$$M^*(f_Z;t) = -\frac{1}{F_Z(t)}\int_0^t f_Z(x)\ln f_Z(x)dx$$

$$= -\frac{1}{F(t/a)}\int_0^t \frac{1}{a} f(x/a) \ln\left\{\frac{1}{a} f(x/a)\right\} dx$$
$$= -\frac{1}{F(t/a)}\int_0^t f(x/a)\{\ln f(x/a) - \ln a\}\, dx$$
$$= \frac{\ln a}{aF(t/a)}\int_0^t f(x/a)\,dx - \frac{1}{aF(t/a)}\int_0^t f(x/a) \ln f(x/a)\, dx$$
$$= \frac{\ln a}{F(t/a)}\int_0^{t/a} f(x)\,dx - \frac{1}{F(t/a)}\int_0^{t/a} f(x) \ln f(x)\, dx$$
$$= \ln a + M^*(f;t).$$

Hence the proof. ■

The following theorem shows that $ICUPL$ class of life distribution is closed under linear transformation.

Theorem 2.3: Let $X \in ICUPL$. Define $Z = aX$, where $a > 0$ is a constant. Then Z $\in ICUPL$.

Proof: The proof follows from Proposition 2.1. ■

Conclusion:

In this paper we introduce a new kind of measure of uncertainty based on conditional expectation. The measure will be particularly useful in the study of ageing pattern of the system. We have establish here a relationship between ageing class $IUCPL$ and a ageing class IUL based on past entropy, which may help different researcher to work on $IUCPL$ class of lifetime distribution. We also have characterize $DRHR$ and $IRHR$ class of life distribution using this new kind of measure of uncertainty based on conditional expectation defined in this paper.

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References:

1. Di Crescenzo, A. and Longobardi, M. (2002): Entropy-based measure of uncertainty in past lifetime distributions. *Journal of Applied Probability*, 39, 434-440.

2. Ebrahimi, N. (1996): How to measure uncertainty about residual lifetime. *Sankhyā* A, 58, 48-57.

3. Ebrahimi, N. and Kirmani, S.N.U.A. (1996*a*): Some results on ordering of survival functions through uncertainty. *Statistics and Probability Letters*, 29, 167-176.

4. Ebrahimi, N. and Kirmani, S.N.U.A. (1996*b*): A characterization of the proportional hazards model through a measure of discrimination between two residual life distributions. *Biometrika*, 83(1), 233-235.

5. Ebrahimi, N. and Pellerey, F. (1995): New partial ordering of survival functions based on notion of uncertainty. *Journal of Applied Probability*, 32, 202-211.

6. Nanda, A.K. and Paul, P. (2006). Some properties of past entropy and their applications. *Metrika*, 64(1), 47-61.

7. Sankaran, P.G. and Gupta, R.P. (1999): Characterization of lifetime distributions using measure of uncertainty. *Calcutta Statistical* Association *Bulletin*. 49(195-196), 159-166.

8. Shannon, C.E. (1948). A mathematical theory of communications. *Bell System Technical Journal*, 27, 379-423.

9. Wiener, N. (1961): **Cybernetics**. (Second edition), MIT Press and John Wiley, New York.